Sydney Technical High School

Name: ....

Maths Class: ....

Year 12
Mathematics Advanced

HSC Course

Assessment 1

December, 2021

Time allowed: 90 minutes

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A reference sheet is provided.

Section 1 Multiple Choice
Questions 1-7
7 Marks

Section II Questions 8-12
65 Marks

Section I 7 Marks

Use multiple choice answer sheet for questions 1-7

Question 1

The derivative of $(x^2-5)^3$ is

(A) $3(x^2-5)^2$

(B) $3(2x-5)^2$

(C) $6x(x^2-5)^2$

(D) $6x(2x-5)^2$

Question 2

What is the value of $\lim\limits_{x\to 2}\frac{x^2-4}{4x-8}$?

(A) 0

(B) 1

(C) 2

(D) 4

Question 3

If $y=e^{x^3}$ then $\frac{dy}{dx}=$

(A) $x^3e^{x^3}$

(B) $3x^2e^{x^3}$

(C) $3e^{x^3}$

(D) $3x^3e^{x^3}$

Question 4

In a raffle, 50 tickets are sold and there is one prize to be won. What is the probability that someone who buys 5 tickets wins the prize?

(A) $\frac{1}{10}$

(B) $\frac{1}{5}$

(C) $\frac{1}{50}$

(D) $\frac{1}{6}$

Question 5

At a particular point on a curve, $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} < 0$. This means the curve is:

(A) Decreasing and is concave down

(B) Decreasing and is concave up

(C) Increasing and is concave down

(D) Increasing and is concave up

Question 6

A particle is moving in a straight line. At time t seconds it has displacement $x = \frac{t}{t+1}$ metres from a fixed point 0 on the line. What is its acceleration $a\ m/s^2$ at time t?

(A) $a = -\frac{2}{(t+1)^3}$

(B) $a = -\frac{1}{(t+1)^2}$

(C) $a = \frac{1}{(t+1)^2}$

(D) $a = \frac{2}{(t+1)^3}$

Question 7

If $a = 10^x$, which of the following is an expression for $\log_{10} \sqrt{a}$?

(A) $\frac{x}{2}$

(B) x

(C) $\sqrt{x}$

(D) x^2

Section II 65 Marks

Questions 8-12
Answer each question in your writing booklet.
Start each question on a NEW sheet of paper.

Question 8 (Start a new page) Mark

a) Evaluate $5e^{-2}$ correct to 2 decimal places 1

b) Differentiate the following with respect to x

(i) $y = 2\sqrt{x}$ 1

(ii) $y = (4 - x)^{-2}$ 1

(iii) $y = e^{x} lnx$ 2

c) In Year 7 at Random High School, every student must do Art or Music. In a group of 100 students surveyed, 47 do Music and 59 do Art.

(i) Draw a Venn diagram to represent the above information. 1

(ii) Find the probability that this student does both Music and Art. 1

(iii) A student is selected at random. Find the probability that this student does Art only. 1

d) Sketch the graph of the curve $y = \log_2(x - 1)$ showing the intercepts on the axes and the equation of the asymptote. 3

e) Simplify fully $\frac{8}{2^{3x} \times 8^{1-x}}$ 2

Question 9 (Start a new page)

a) Differentiate $\frac{5x^2-2x}{x}$ 1

b) Use the differentiating by first principles formula 3

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to find $f'(x)$ when $f(x) = x^2 - 3$

c) Find the equation of the normal to the curve $y = ln(2x)$ at the point where $x = 2$. 3

d) Solve the equation 3

$$\log_2(x-1) - \log_2(x+1) = 2$$

and check your answer(s).

e) Find the coordinates of the stationary point on the curve $y = x^2 - \frac{16}{x}$ 3

Question 10 (Start a new page)

a) The displacement x m from the origin at time t seconds of a particle travelling in a straight line is given by $x = t^3 - 9t$, where $t \geq 0$.

(i) Calculate the velocity of the particle, when $t = 2$. 1

(ii) Find when the particle is stationary. 2

(iii) Show that the particle is always accelerating in the positive direction. 2

(iv) Calculate how far the particle travels during the 3rd second. 2

b) James has 3 tickets in a raffle in which there are 30 tickets and 2 prizes.

(i) Complete the probability tree diagram to represent the above information. 1

(ii) Find the probability that James wins the second prize. 2

(iii) Find the probability that he wins at least one prize. 1

c) Two dice are rolled. Given that the total is greater than 7, find the probability the total is 10. 2

Question 11 (Start a new page)

a) Solve $e^{2x} - 2e^{x} = 0$ 2

b) (i) Differentiate $y = 2^{x}$ 1

(ii) Find the equation of the tangent to this curve at $x = 1$. 2

(iii) Find the angle of inclination of this tangent to the x axis rounded to the nearest degree. 1

c) Consider the curve $y = x^4 - 4x^3$

(i) Find any stationary points and determine their nature. 3

(ii) Sketch the curve showing all important features. 2

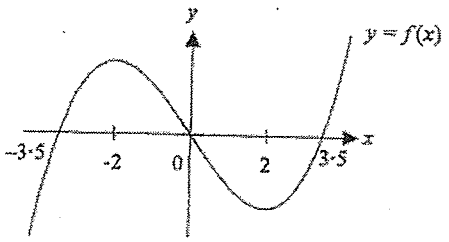
(iii) Find the domain where the curve is concave down. 1

(iv) Find the global minimum over the domain [-1,4]. 1

Question 12 (Start a new page)

a) Use the chain rule or otherwise to find $\frac{dy}{dx}$ when $y = \log_2(\log_e x)$ 2

b) 2



Sketch the curve $y = f'(x)$ showing the shape of the curve and the x intercepts.

c) A fair dice is rolled once. "A" is the event where the score is even and "B" is the event the score is 5 or 6. Determine, giving a reason, whether or not the events A and B are independent. 2

d) Differentiate $y = ln\ \left(\frac{2x+3}{3x-1}\right)$ 1

e) Hot tea is poured into a cup. The temperature of tea can be modelled by $\boldsymbol{T = 25 + 70(e)^{-0.4t}}$ where $\boldsymbol{T}$ is the temperature of the tea in degrees Celsius, $\boldsymbol{t}$ minutes after it is poured.

(i) What is the temperature of the tea 4 minutes after it has been poured? 1

(ii) At what rate is the tea cooling 4 minutes after it has been poured? 2

(iii) How long after the tea is poured will it take for its temperature to reach 55°C? 3

END OF PAPER

Student Name: ____________________ Teacher Name: ____________________

Solutions to 2021 Adv. Task 1

1. $3(x^2-5)^2 \times 2x$

$6x(x^2-5)^2$

C

2. $\lim_{x \to 2} \frac{x^2-4}{4x-8}$

$\lim_{x \to 2} \frac{(x-2)(x+2)}{4(x-2)}$

1

B

3. $\frac{d}{dx}(e^{x^3})$

$= 3x^2 e^{x^3}$

B

4. $\frac{5}{50}$

$= \frac{1}{10}$

A

5. C

6. $x = \frac{t}{t+1}$

$v = \frac{(t+1) \times 1 - t}{(t+1)^2}$

$v = (t+1)^{-2}$

$a = -2(t+1)^{-3}$

$= \frac{-2}{(t+1)^3}$

A

7. $\log_{10} \sqrt{a}$

$\log_{10} a^{\frac{1}{2}}$

$\frac{1}{2} \log_{10} a$ (Log law)

but $a = 10^x$

$\therefore \log_{10} a = x$

$= \frac{1}{2} \times x$

A

8a) $5e^{-2}$

0.68

b) (i) $y = 2\sqrt{x}$

$y = 2x^{\frac{1}{2}}$

$\frac{dy}{dx} = 1x^{-\frac{1}{2}}$

$= \frac{1}{\sqrt{x}}$

(ii) $y = (4-x)^{-2}$

$\frac{dy}{dx} = -2(4-x)^{-3} \times -1$

$= \frac{2}{(4-x)^3}$

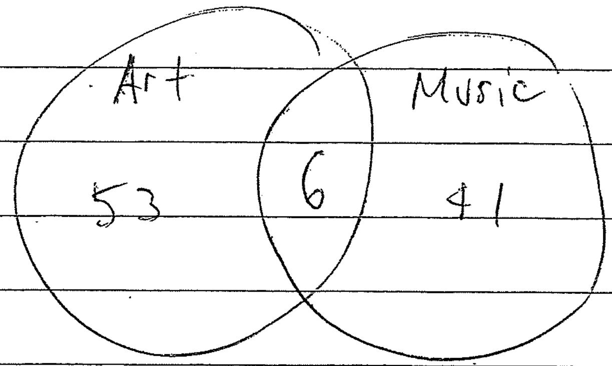
(iii) $y = e^x \ln x$

$\frac{dy}{dx} = e^x \ln x + e^x \times \frac{1}{x}$

$= e^x \ln x + \frac{e^x}{x}$

Student Name: ______________________ Teacher Name: ______________________

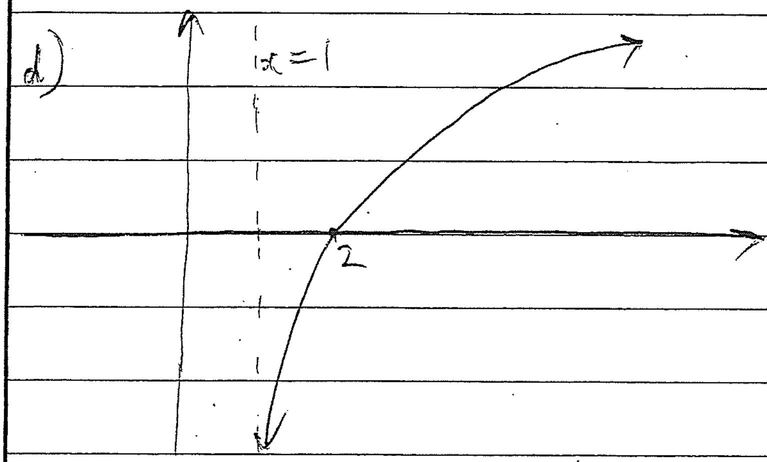
c) (i)



(ii) $P(\text{Music} + \text{Art}) = \frac{6}{100}$ or $\frac{3}{50}$

(iii) $P(\text{Art only}) = \frac{53}{100}$

d)



e) $\frac{8}{2^{3x} \times 8^{1-x}}$

$\frac{2^3}{2^{3x} \times (2^3)^{1-x}}$

$\frac{2^3}{2^{3x} \times 2^{3-3x}}$

$2^{\cancel{3}-\cancel{3x}-\cancel{3}+\cancel{3x}}$

$= 1$

9a) $\frac{5x^2 - 2x}{x}$

$= 5x - 2$

$\frac{d}{dx} = 5$

b) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

When $f(x) = x^2 - 3$

$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - 3 - (x^2 - 3)}{h}$

$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{-3} \cancel{-x^2} \cancel{+3}}{h}$

$= \lim_{h \to 0} 2x + h$

$= 2x$

Student Name: ______________________ Teacher Name: ______________________

c) $y = \ln(2x)$

$\frac{dy}{dx} = \frac{2}{2x}$

$= \frac{1}{x}$

when $x = 2$ $\frac{dy}{dx} = \frac{1}{2}$

$\therefore$ gradient of normal $= -2$

Point is $(2, \ln 4)$

$y - \ln 4 = -2(x-2)$

$y - \ln 4 = -2x + 4$

$y = -2x + \ln 4 + 4$

d) $\log_2(x-1) - \log_2(x+1) = 2$

$\log_2\left(\frac{x-1}{x+1}\right) = 2$

$2^2 = \frac{x-1}{x+1}$

$4x + 4 = x - 1$

$3x = -5$

$x = \frac{-5}{3}$

No solution as $(\frac{-5}{3} - 1) < 0$ and you can't log a negative

e) $y = x^2 - \frac{16}{x}$

$\frac{dy}{dx} = 2x + \frac{16}{x^2} = 0$

for stationary pts.

$2x^3 + 16 = 0$

$x^3 + 8 = 0$

$x = -2 \Rightarrow y = 4 - \frac{16}{-2} = 12$

$(-2, 12)$ is stat. pt.

10.(i) $x = t^3 - 9t$

$v = \frac{dx}{dt} = 3t^2 - 9$

when $t = 2$

$v = 3 \times 2^2 - 9$

$= 3$ m/s

(ii) Stationary when

$v = 0$

$3t^2 - 9 = 0$

$3t^2 = 9$

$t^2 = 3$

$t = \sqrt{3}$ $(t > 0)$

Student Name: ______________________ Teacher Name: ______________________

(iii) $a = \frac{dv}{dt} = 6t$

since $t > 0$, $a > 0$

$\therefore$ accelerates in positive direction

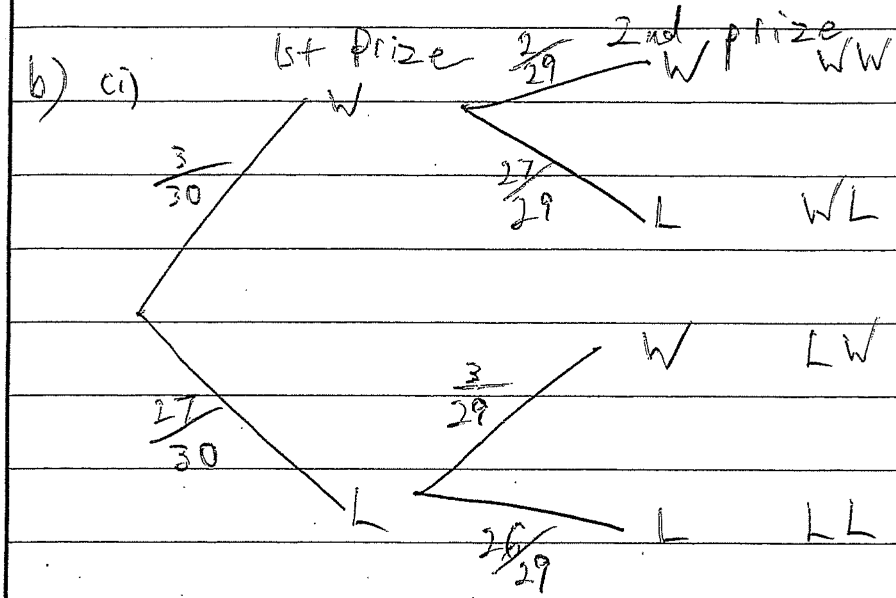
(iv) 3rd second is from $t=2$ to $t=3$

at $t=2$, $x = -10$

at $t=3$, $x = 0$

$\therefore$ travelled 10m

b) (i)



(ii) P(wins 2nd) is WW + LW

$\frac{3}{30} \times \frac{2}{29} + \frac{27}{30} \times \frac{3}{29} = \frac{1}{10}$

(iii) P(at least one prize) = 1 − P(No prizes)

$= 1 - \frac{27}{30} \times \frac{26}{29}$

$= \frac{28}{145}$

c)

11	21	31	41	51	61
12	22	32	42	52	62
13	23	33	43	53	63
14	24	34	44	54	64
15	25	35	45	55	65
16	26	36	46	56	66

15 totals > 7

$P(10) = \frac{3}{15}$ or $\frac{1}{5}$

Student Name: ______________________ Teacher Name: ______________________

11 a) $e^{2x} - 2e^{x} = 0$

$e^{x}(e^{x} - 2) = 0$

$e^{x} = 0$ or $e^{x} = 2$

No sol'n or $x = \ln 2$

$x = \log_e 2$ only solution

b) (i) $y = 2^{x}$

$\frac{dy}{dx} = 2^{x} \log_e 2$

(ii) Sub $x = 1$ into $\frac{dy}{dx}$

$m = 2\ln 2$ gradient of tangent at $(1,2)$

$y - 2 = 2\ln 2(x - 1)$

$y = 2\ln 2 x + 2 - 2\ln 2$

(iii) $m = \tan\theta$

$2\ln 2 = \tan\theta$

$\theta = \tan^{-1}(2\ln 2)$

$\theta = 54^{\circ}$

c) $y = x^{4} - 4x^{3}$

$\frac{dy}{dx} = 4x^{3} - 12x^{2} = 0$ for stationary points

$4x^{2}(x - 3) = 0$

$x = 0$ or 3

$\frac{d^2y}{dx^2} = 12x^{2} - 24x$

When $x = 0$, $\frac{d^2y}{dx^2} = 0$

Further test.

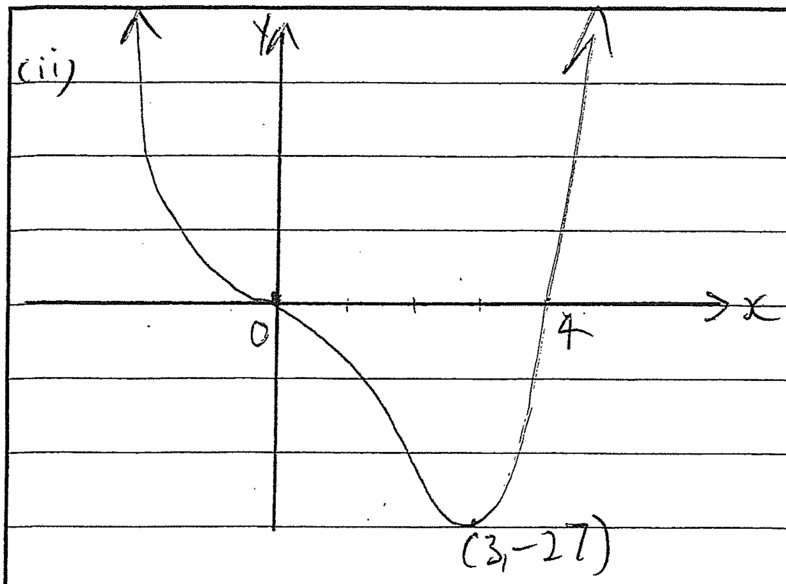
x	-1	0	1
$\frac{dy}{dx}$	-16	0	-8

$\therefore$ Horizontal inflexion at $(0, 0)$

When $x = 3$, $\frac{d^2y}{dx^2} > 0$ $\therefore$ a minimum at $(3, -27)$

Student Name: ______________________ Teacher Name: ______________________

(ii)



(iii) Concave down when $\frac{d^2y}{dx^2} < 0$

$12x^2 - 24x < 0$

$12x(x - 24) < 0$

$0 < x < 2$

(iv) Global minimum is -27.

12 a) $y = \log_2(\log_e x)$

Let $v = \log_e x$

$\frac{dv}{dx} = \frac{1}{x}$

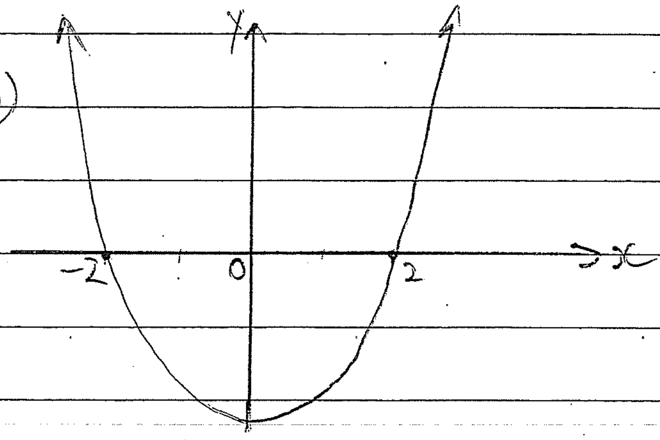
$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{dx}$

$y = \log_2 v$ so

$\frac{dy}{dx} = \frac{1}{\log_e 2 \times v} \times \frac{1}{x}$

$= \frac{1}{x \log_e 2 \log_e x}$

b)



c) $P(A)P(B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} = P(A \cap B)$

Hence A and B are independent

d) $y = \ln\left(\frac{2x+3}{3x-1}\right)$

$= \ln(2x+3) - \ln(3x-1)$

$\frac{dy}{dx} = \frac{2}{2x+3} - \frac{3}{3x-1}$

Student Name: ______________________ Teacher Name: ______________________

e (i) $T = 25 + 70e^{-0.4t}$

when $t = 4$

$T = 25 + 70e^{-1.6}$

$T = 39^{\circ}$ to nearest degree

(ii) $\frac{dT}{dt} = 70 \times -0.4e^{-0.4t}$

At $t = 4$

$\frac{dT}{dt} = 70 \times -0.4e^{-0.4 \times 4}$

$= -5.7^{\circ}C/\text{minute}$

(iii) $55 = 25 + 70e^{-0.4t}$

$30 = 70e^{-0.4t}$

$\frac{3}{7} = e^{-0.4t}$

$\log_e \frac{3}{7} = -0.4t$

$t = \frac{\log_e \frac{3}{7}}{-0.4}$

$= 2.1$ minutes.

Year 12 Task 1 Advanced comments

Question 8

b)(iii) Quite a few students made errors with this question. Many failed to realise that, to differentiate this, the product rule was needed.

c)(i) There were many errors with the drawing of the Venn diagram. Students made simple errors here and failed to write the correct numbers for music, art and the intersection.

d) This question was poorly done. Many students failed to draw the correct shape of the graph and failed to identify what type of translation was occurring here. (i.e. the direction of the translation).

e) This question was poorly done. Students failed to, initially, rewrite 8 as 2^3. If they did not do this, many issues with simplification arose. Students also failed to use basic index laws correctly.

Question 9

a) For one mark the quotient rule was unnecessary – you could have simplified first and made the question one marks worth of work
b) Setting out was terrible, must revise first principles and set your work out carefully
c) Many found the equation of the tangent instead of the normal, some forgot to substitute the value of x in for the gradient OR did not give the corresponding y-value (it is not automatically 0)
d) Lots of algebraic mistakes and incorrect application of log laws. Many did not test their solution OR worse, told me that the log(-8/3) could be evaluated. The restriction here was x>1 not x>0.
e) Did not require the nature of the stationary point – time wasted. Care needed in the algebra.

Question 10

a)(ii) Some students incorrectly solved $x = 0$. The particle is stationary when $v = 0$. Also, you must dismiss the negative answer (and should write $t > 0$) as time must be positive.

a)(iv) Few students in the cohort got this correct. During the 3rd second means you must find the distance travelled from $t = 2$ to $t = 3$. You were not asked to simply find the displacement at $t = 3$.

b)(i) Many issues drawing the correct tree diagram. For James, the two options for prize one were to win or lose, and the two options for prize two were to win or lose. Setting this situation up correctly led to success with the fractions on the branches.

b)(ii) You cannot simply add 2/29 and 3/29 to find the probability of winning the second prize. To win the second prize, James had to either win both prizes, or lose the first and win the second.

(iii) Most success was students using the complementary event, $1 - P(wins\ no\ prizes)$

c) Those who drew the full array diagram had more success than those who did not. 'Total greater than 7', does NOT include 7, the sum must be 8 or more giving a reduced sample space of 15. Out of this reduced sample space, 3 total to 10.